Solving a Two-Body Dynamics Problem using 6-D Vectors

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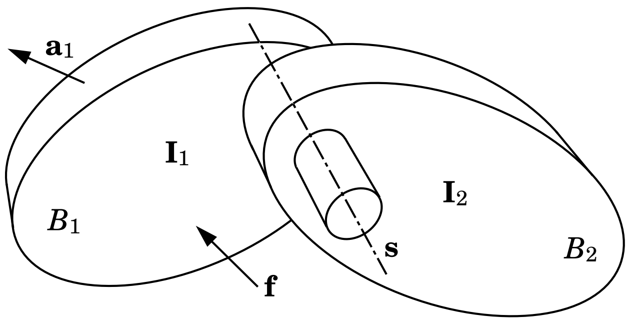
Problem Statement

A rigid-body system, comprising two bodies B_1 and B_2 connected by a revolute joint, is initially at rest. A force $\mathbf{f}$ is applied to B_1 causing both bodies to accelerate. Given that the two bodies have inertias $\mathbf{I}_1$ and $\mathbf{I}_2$, respectively, and that the joint's axis of motion is defined by the vector $\mathbf{s}$, calculate the acceleration of B_1 as a function of $\mathbf{f}$.

Diagram



Solution

Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the net forces acting on B_1 and B_2, respectively, and let $\mathbf{a}_1$ and $\mathbf{a}_2$ be their respective accelerations, then their equations of motion are

$$\mathbf{f}_1 = \mathbf{I}_1\,\mathbf{a}_1 \tag{1}$$

and

$$\mathbf{f}_2 = \mathbf{I}_2\,\mathbf{a}_2 \tag{2}$$

(given that the velocities are zero). $\mathbf{f}$ is the total net force acting on the system, so

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2\,. \tag{3}$$

$\mathbf{f}$ acts only on B_1, so $\mathbf{f}_2$ must be transmitted from B_1 to B_2 via the joint. If we can express $\mathbf{f}_2$ in terms of $\mathbf{a}_1$ then the problem will be solved.

The joint permits B_2 to accelerate relative to B_1 in the direction specified by $\mathbf{s}$, so $\mathbf{a}_2$ may be expressed in the form

$$\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{s}\,\alpha \tag{4}$$

where α is a scalar representing the joint acceleration. The joint will also transmit any nonworking constraint force between the two bodies, i.e., any force $\mathbf{f}_c$ satisfying $\mathbf{s}^T\,\mathbf{f}_c = 0$. We know that $\mathbf{f}_2$ is transmitted to B_2 through the joint, so it must satisfy

$$\mathbf{s}^T\,\mathbf{f}_2 = 0. \tag{5}$$

Substituting Eq. 4 into Eq. 2 gives

$$\mathbf{f}_2 = \mathbf{I}_2\,(\mathbf{a}_1 + \mathbf{s}\,\alpha), \tag{6}$$

and substituting this equation into Eq. 5 gives

$$\mathbf{s}^T\,\mathbf{I}_2\,(\mathbf{a}_1 + \mathbf{s}\,\alpha) = 0,$$

from which we get the following expression for α:

$$\alpha = -\frac{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{a}_1}{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{s}}. \tag{7}$$

Substituting Eq. 7 into Eq. 6 gives

$$\mathbf{f}_2 = \mathbf{I}_2\left(\mathbf{a}_1 - \frac{\mathbf{s}\,\mathbf{s}^T\,\mathbf{I}_2}{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{s}}\,\mathbf{a}_1\right),$$

and substituting this equation, together with Eq. 1, back into Eq. 3 gives

$$\begin{aligned}\mathbf{f} &= \mathbf{I}_1\,\mathbf{a}_1 + \mathbf{I}_2\,\mathbf{a}_1 - \frac{\mathbf{I}_2\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{I}_2}{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{s}}\,\mathbf{a}_1 \\ &= \left(\mathbf{I}_1 + \mathbf{I}_2 - \frac{\mathbf{I}_2\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{I}_2}{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{s}}\right)\mathbf{a}_1.\end{aligned}$$

The expression in brackets is non-singular, and may be inverted to express $\mathbf{a}_1$ in terms of $\mathbf{f}$:

$$\mathbf{a}_1 = \left(\mathbf{I}_1 + \mathbf{I}_2 - \frac{\mathbf{I}_2\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{I}_2}{\mathbf{s}^T\,\mathbf{I}_2\,\mathbf{s}}\right)^{-1}\mathbf{f}. \tag{8}$$

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