Solving a Two-Body Dynamics Problem using 3-D Vectors

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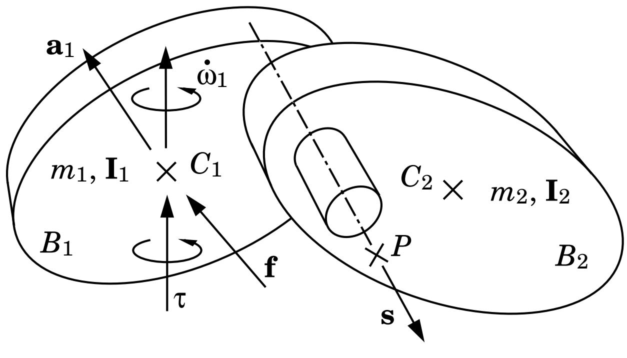
Problem Statement

A rigid-body system, comprising two bodies B_1 and B_2 connected by a revolute joint, is initially at rest. A system of forces acts on B_1 causing both bodies to accelerate. B_1 is characterized by its mass, m_1, the location of its centre of mass, C_1, and its rotational inertia about its centre of mass, $\mathbf{I}_1$. B_2 is similarly characterized by m_2, C_2 and $\mathbf{I}_2$. The joint's axis of rotation passes through the point P in the direction given by the vector $\mathbf{s}$. The resultant of the applied system of forces is given by a force $\mathbf{f}$ acting through the centre of mass of B_1 together with a couple $\boldsymbol{\tau}$, and the resulting acceleration of B_1 is given by the acceleration $\mathbf{a}_1$ of its centre of mass and its angular acceleration $\dot{\boldsymbol{\omega}}_1$. The problem is to express $\mathbf{a}_1$ and $\dot{\boldsymbol{\omega}}_1$ in terms of $\mathbf{f}$ and $\boldsymbol{\tau}$.

Diagram



Solution

The key to solving a problem like this is to realise that the joint allows one degree of motion freedom between the two bodies, but also imposes one constraint on the forces that can be transmitted through the joint. The latter can be used to eliminate the former, leading (eventually) to a system of equations involving only the applied forces and the acceleration of B_1. Key steps in the solution procedure include: expressing the acceleration of B_2 in terms of the acceleration of B_1 and the joint acceleration; expressing the force transmitted through the joint as a function of the acceleration of B_2; and using the joint force constraint equation to solve for the joint acceleration. Once this is accomplished, every unknown in the system can be expressed as a function of the acceleration of B_1.

Let us introduce the following quantities. Let $\mathbf{f}_1$, $\boldsymbol{\tau}_1$, $\mathbf{f}_2$ and $\boldsymbol{\tau}_2$ be the net forces and couples acting on B_1 and B_2, respectively, where the lines of action of $\mathbf{f}_1$ and $\mathbf{f}_2$ pass through C_1 and C_2, respectively; let $\mathbf{a}_2$ and $\dot{\boldsymbol{\omega}}_2$ be the linear and angular acceleration of B_2, expressed at its centre of mass; let ${}^P\mathbf{a}_1$, ${}^P\dot{\boldsymbol{\omega}}_1$, ${}^P\mathbf{a}_2$ and ${}^P\dot{\boldsymbol{\omega}}_2$ be the linear and angular accelerations of B_1 and B_2 expressed at P; and let ${}^P\mathbf{f}_2$ and ${}^P\boldsymbol{\tau}_2$ be the net force and couple acting on B_2 expressed at P. As the system of applied forces acts only on B_1, the net force and couple acting on B_2 are also the net force and couple transmitted through the joint. Let us also define $\mathbf{r}_1 = \overrightarrow{C_1P}$ and $\mathbf{r}_2 = \overrightarrow{C_2P}$, and let α be the joint acceleration variable.

The equations of motion of the two bodies, expressed at their centres of mass, are

$$\mathbf{f}_1 = m_1\,\mathbf{a}_1, \tag{1}$$

$$\boldsymbol{\tau}_1 = \mathbf{I}_1\,\dot{\boldsymbol{\omega}}_1, \tag{2}$$

$$\mathbf{f}_2 = m_2\,\mathbf{a}_2 \tag{3}$$

and

$$\boldsymbol{\tau}_2 = \mathbf{I}_2\,\dot{\boldsymbol{\omega}}_2. \tag{4}$$

(There are no velocity terms because the bodies are initially at rest, so their linear and angular velocities are zero.)

The rules for transferring forces and accelerations (of bodies at rest) from one point to another provide us with the following relationships

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between quantities referred to C_1, C_2 and P:

$$ {}^{P}\mathbf{a}_1 = \mathbf{a}_1 - \mathbf{r}_1 \times \dot{\boldsymbol{\omega}}_1, \tag{5} $$

$$ {}^{P}\mathbf{a}_2 = \mathbf{a}_2 - \mathbf{r}_2 \times \dot{\boldsymbol{\omega}}_2, \tag{6} $$

$$ {}^{P}\dot{\boldsymbol{\omega}}_1 = \dot{\boldsymbol{\omega}}_1, \tag{7} $$

$$ {}^{P}\dot{\boldsymbol{\omega}}_2 = \dot{\boldsymbol{\omega}}_2, \tag{8} $$

$$ {}^{P}\mathbf{f}_2 = \mathbf{f}_2 \tag{9} $$

and

$$ {}^{P}\boldsymbol{\tau}_2 = \boldsymbol{\tau}_2 - \mathbf{r}_2 \times \mathbf{f}_2. \tag{10} $$

$\mathbf{f}$ and $\boldsymbol{\tau}$ are the total net force and couple acting on the system, so they must be equal to the sum of the net forces and couples acting on the individual bodies. Expressed at C_1, the equations are

$$ \mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 \tag{11} $$

and

$$ \boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{f}_2. \tag{12} $$

And finally, the joint allows B_2 one degree of motion freedom relative to B_1 and imposes one constraint on the couple transmitted from B_1 to B_2. Expressed at P, the equations are

$$ {}^{P}\mathbf{a}_2 = {}^{P}\mathbf{a}_1, \tag{13} $$

$$ {}^{P}\dot{\boldsymbol{\omega}}_2 = {}^{P}\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha \tag{14} $$

and

$$ \mathbf{s}^T\, {}^{P}\boldsymbol{\tau}_2 = 0. \tag{15} $$

(There is no constraint on ${}^{P}\mathbf{f}_2$. Eq. 15 is sufficient to ensure that the force and couple transmitted by the joint perform no work in the direction of relative motion permitted by the joint.)

We are now ready to solve the problem. Let us start by calculating $\mathbf{a}_2$ and $\dot{\boldsymbol{\omega}}_2$ in terms of $\mathbf{a}_1$, $\dot{\boldsymbol{\omega}}_1$ and α. From Eqs. 8, 14 and 7 we have

$$ \begin{aligned} \dot{\boldsymbol{\omega}}_2 &= {}^{P}\dot{\boldsymbol{\omega}}_2 \\ &= {}^{P}\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha \\ &= \dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha, \end{aligned} \tag{16} $$

and from Eqs. 6, 13, 5 and 16 we have

$$ \begin{aligned} \mathbf{a}_2 &= {}^{P}\mathbf{a}_2 + \mathbf{r}_2 \times \dot{\boldsymbol{\omega}}_2 \\ &= {}^{P}\mathbf{a}_1 + \mathbf{r}_2 \times (\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha) \\ &= \mathbf{a}_1 - \mathbf{r}_1 \times \dot{\boldsymbol{\omega}}_1 + \mathbf{r}_2 \times (\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha) \\ &= \mathbf{a}_1 + (\mathbf{r}_2 - \mathbf{r}_1) \times \dot{\boldsymbol{\omega}}_1 + \mathbf{r}_2 \times \mathbf{s}\,\alpha. \end{aligned} \tag{17} $$

Now let us calculate α. From Eqs. 15, 10, 3, 4, 16 and 17 we get

$$ \begin{aligned} 0 &= \mathbf{s}^T\, {}^{P}\boldsymbol{\tau}_2 \\ &= \mathbf{s}^T(\boldsymbol{\tau}_2 - \mathbf{r}_2 \times \mathbf{f}_2) \\ &= \mathbf{s}^T(\mathbf{I}_2\,\dot{\boldsymbol{\omega}}_2 - m_2\,\mathbf{r}_2 \times \mathbf{a}_2) \\ &= \mathbf{s}^T(\mathbf{I}_2\,(\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha) - m_2\,\mathbf{r}_2 \times \\ &\qquad (\mathbf{a}_1 + (\mathbf{r}_2 - \mathbf{r}_1) \times \dot{\boldsymbol{\omega}}_1 + \mathbf{r}_2 \times \mathbf{s}\,\alpha)). \end{aligned} $$

Collecting terms in α gives

$$ \begin{aligned} &\mathbf{s}^T(\mathbf{I}_2\,\mathbf{s} - m_2\,\mathbf{r}_2 \times (\mathbf{r}_2 \times \mathbf{s}))\,\alpha + \mathbf{s}^T(\mathbf{I}_2\,\dot{\boldsymbol{\omega}}_1 \\ &\qquad - m_2\,\mathbf{r}_2 \times (\mathbf{a}_1 + (\mathbf{r}_2 - \mathbf{r}_1) \times \dot{\boldsymbol{\omega}}_1)) = 0, \end{aligned} $$

hence

$$ \alpha = -\frac{\mathbf{s}^T(\mathbf{I}_2\,\dot{\boldsymbol{\omega}}_1 - m_2\,\mathbf{r}_2 \times (\mathbf{a}_1 + (\mathbf{r}_2 - \mathbf{r}_1) \times \dot{\boldsymbol{\omega}}_1))}{\mathbf{s}^T(\mathbf{I}_2\,\mathbf{s} - m_2\,\mathbf{r}_2 \times (\mathbf{r}_2 \times \mathbf{s}))} \tag{18} $$

This equation is only valid if the denominator on the RHS is not equal to zero, so we must investigate the necessary conditions for it to be nonzero. This problem can be solved using the following trick. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, the cross product $\mathbf{u} \times \mathbf{v}$ can be expressed in the form $\mathbf{u} \times \mathbf{v} = \tilde{\mathbf{u}}\,\mathbf{v}$, where $\tilde{\mathbf{u}}$ is the skew-symmetric matrix

$$ \tilde{\mathbf{u}} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}. $$

Using this trick, we can express the denominator in the form $\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}$ where

$$ \begin{aligned} \mathbf{J} &= \mathbf{I}_2 - m_2\,\tilde{\mathbf{r}}_2\,\tilde{\mathbf{r}}_2 \\ &= \mathbf{I}_2 + m_2\,\tilde{\mathbf{r}}_2^T\,\tilde{\mathbf{r}}_2. \end{aligned} \tag{19} $$

$\mathbf{J}$ is therefore the sum of an SPD matrix and an SPSD matrix, hence itself also SPD, so the denominator of Eq. 18 is guaranteed to be strictly greater than zero. Substituting Eq. 19 into Eq. 18 gives us the following simplified expression for α:

$$ \alpha = -\frac{\mathbf{s}^T(\mathbf{J}\,\dot{\boldsymbol{\omega}}_1 - m_2\,\mathbf{r}_2 \times (\mathbf{a}_1 - \mathbf{r}_1 \times \dot{\boldsymbol{\omega}}_1))}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}} \tag{20} $$

The next step is to express $\mathbf{f}$ and $\boldsymbol{\tau}$ in terms of $\mathbf{a}_1$, $\dot{\boldsymbol{\omega}}_1$ and α, and then to eliminate α using Eq. 20. Let us start with $\mathbf{f}$. From Eqs. 11, 1, 3 and 17 we get

$$ \mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 $$

$$
\begin{aligned}
&= m_1\,\mathbf{a}_1 + m_2\,\mathbf{a}_2 \\
&= m_1\,\mathbf{a}_1 + m_2(\mathbf{a}_1 + (\mathbf{r}_2 - \mathbf{r}_1)\times\dot{\boldsymbol{\omega}}_1 + \mathbf{r}_2\times\mathbf{s}\,\alpha) \\
&= (m_1+m_2)\mathbf{a}_1 + m_2(\mathbf{r}_2-\mathbf{r}_1)\times\dot{\boldsymbol{\omega}}_1 \\
&\quad + m_2\,\mathbf{r}_2\times\mathbf{s}\,\alpha.
\end{aligned}
$$

Elimitating α using Eq. 20 gives

$$
\mathbf{f} = (m_1+m_2)\mathbf{a}_1 + m_2(\mathbf{r}_2-\mathbf{r}_1)\times\dot{\boldsymbol{\omega}}_1 - m_2\,\frac{\mathbf{r}_2\times\mathbf{s}\,\mathbf{s}^T(\mathbf{J}\,\dot{\boldsymbol{\omega}}_1 - m_2\,\mathbf{r}_2\times(\mathbf{a}_1 - \mathbf{r}_1\times\dot{\boldsymbol{\omega}}_1))}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}};
$$

and collecting terms in $\mathbf{a}_1$ and $\dot{\boldsymbol{\omega}}_1$ gives

$$
\begin{aligned}
\mathbf{f} &= \left(m_1 + m_2 + m_2^2\,\frac{\tilde{\mathbf{r}}_2\,\mathbf{s}\,\mathbf{s}^T\,\tilde{\mathbf{r}}_2}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}\right)\mathbf{a}_1 + \\
&\left(m_2(\tilde{\mathbf{r}}_2-\tilde{\mathbf{r}}_1) - m_2\,\frac{\tilde{\mathbf{r}}_2\,\mathbf{s}\,\mathbf{s}^T(\mathbf{J} + m_2\,\tilde{\mathbf{r}}_2\,\tilde{\mathbf{r}}_1)}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}\right)\dot{\boldsymbol{\omega}}_1.
\end{aligned} \tag{21}
$$

Repeating the procedure for $\boldsymbol{\tau}$, Eqs. 12, 2, 3, 4, 16 and 17 give

$$
\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + (\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{f}_2 \\
&= \mathbf{I}_1\,\dot{\boldsymbol{\omega}}_1 + \mathbf{I}_2\,\dot{\boldsymbol{\omega}}_2 + m_2(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{a}_2 \\
&= \mathbf{I}_1\,\dot{\boldsymbol{\omega}}_1 + \mathbf{I}_2(\dot{\boldsymbol{\omega}}_1 + \mathbf{s}\,\alpha) + m_2(\mathbf{r}_1-\mathbf{r}_2)\times \\
&\quad (\mathbf{a}_1 + (\mathbf{r}_2-\mathbf{r}_1)\times\dot{\boldsymbol{\omega}}_1 + \mathbf{r}_2\times\mathbf{s}\,\alpha) \\
&= (\mathbf{I}_1 + \mathbf{I}_2 - m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)^2)\dot{\boldsymbol{\omega}}_1 \\
&\quad + m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)\mathbf{a}_1 + \mathbf{K}\,\mathbf{s}\,\alpha,
\end{aligned} \tag{22}
$$

where

$$
\begin{aligned}
\mathbf{K} &= \mathbf{I}_2 + m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)\tilde{\mathbf{r}}_2 \\
&= \mathbf{J} + m_2\,\tilde{\mathbf{r}}_1\,\tilde{\mathbf{r}}_2.
\end{aligned} \tag{23}
$$

Note that Eq. 20 can now be simplified to

$$
\alpha = -\frac{\mathbf{s}^T(\mathbf{K}^T\,\dot{\boldsymbol{\omega}}_1 - m_2\,\tilde{\mathbf{r}}_2\,\mathbf{a}_1)}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}. \tag{24}
$$

Eliminating α from Eq. 22 using Eq. 24 gives

$$
\begin{aligned}
\boldsymbol{\tau} &= (\mathbf{I}_1 + \mathbf{I}_2 - m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)^2)\dot{\boldsymbol{\omega}}_1 + \\
&\quad m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)\mathbf{a}_1 - \frac{\mathbf{K}\,\mathbf{s}\,\mathbf{s}^T(\mathbf{K}^T\,\dot{\boldsymbol{\omega}}_1 - m_2\,\tilde{\mathbf{r}}_2\,\mathbf{a}_1)}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}},
\end{aligned}
$$

and collecting terms in $\dot{\boldsymbol{\omega}}_1$ and $\mathbf{a}_1$ gives

$$
\begin{aligned}
\boldsymbol{\tau} &= \left(\mathbf{I}_1 + \mathbf{I}_2 - m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)^2 - \frac{\mathbf{K}\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{K}^T}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}\right)\dot{\boldsymbol{\omega}}_1 \\
&\quad + \left(m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2) + m_2\,\frac{\mathbf{K}\,\mathbf{s}\,\mathbf{s}^T\tilde{\mathbf{r}}_2}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}\right)\mathbf{a}_1.
\end{aligned} \tag{25}
$$

The final step is to combine Eqs. 21 and 25 into a single equation:

$$
\begin{bmatrix} \mathbf{f} \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \dot{\boldsymbol{\omega}}_1 \end{bmatrix},
$$

where

$$
\mathbf{A} = m_1 + m_2 + m_2^2\,\frac{\tilde{\mathbf{r}}_2\,\mathbf{s}\,\mathbf{s}^T\,\tilde{\mathbf{r}}_2}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}},
$$

$$
\mathbf{B} = m_2(\tilde{\mathbf{r}}_2-\tilde{\mathbf{r}}_1) - m_2\,\frac{\tilde{\mathbf{r}}_2\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{K}^T}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}},
$$

$$
\mathbf{C} = m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2) + m_2\,\frac{\mathbf{K}\,\mathbf{s}\,\mathbf{s}^T\tilde{\mathbf{r}}_2}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}
$$

and

$$
\mathbf{D} = \mathbf{I}_1 + \mathbf{I}_2 - m_2(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_2)^2 - \frac{\mathbf{K}\,\mathbf{s}\,\mathbf{s}^T\,\mathbf{K}^T}{\mathbf{s}^T\,\mathbf{J}\,\mathbf{s}}.
$$

(Notice that $\mathbf{A}$ and $\mathbf{D}$ are symmetric, and that $\mathbf{B} = \mathbf{C}^T$.) The solution to the original problem is then

$$
\begin{bmatrix} \mathbf{a}_1 \\ \dot{\boldsymbol{\omega}}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\tau} \end{bmatrix}. \tag{26}
$$

At this point, we should prove that the 6×6 coefficient matrix is nonsingular. It is in fact an SPD matrix, but the easiest way to prove it is to show that it is identical to the solution obtained using the 6-D vector approach, which is easily shown to be an SPD matrix.